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ENGINEERING MATHEMATICS | STUDY NOTES

# UNIT I — PROPERTIES OF TRIANGLE

Sine Rule | Cosine Rule | Projection Formula

This unit builds three core tools for solving *any* triangle — not just right-angled ones — each developed from first principles with a full proof, then applied to worked engineering problems.

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BY THE END OF THIS UNIT, YOU WILL BE ABLE TO

- Explain the **Sine Formula**, with proof.
- Explain the **Cosine Formula**, with proof.
- Explain the **Projection Formula**, with proof.
- Apply knowledge of the sine, cosine, and projection formulas to solve engineering problems.
- Explain the concept of **similarity**, and use theorems involving similarity.

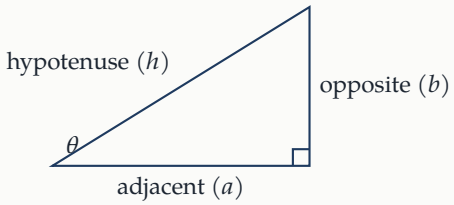
### Concept of Similarity

WHAT IS SIMILARITY?

Two triangles are **similar** when they have exactly the same shape but not necessarily the same size — their corresponding angles are equal, and their corresponding sides are all in the same ratio to one another. Similarity is what makes it possible to relate measurements inside a triangle (like the altitude used throughout this unit) back to its sides and angles, which is the foundation the Sine Rule, Cosine Rule, and Projection Formula are all built on.

## Foundations: The Right Triangle

Every relation in this unit is built on the right triangle and the Pythagorean theorem. Before the Sine and Cosine Rules can make sense for *any* triangle, you first need to be completely comfortable with how the three sides of a **right-angled** triangle relate to one of its angles, $\theta$:




*Right triangle with reference angle $\theta$: hypotenuse $h$, opposite side $b$, adjacent side $a$.*

HOW TO READ THIS DIAGRAM

- **Hypotenuse** ($h$) — the longest side, always opposite the right angle (the 90° corner). It never touches $\theta$ directly except at one end.
- **Opposite** ($b$) — the side directly across from angle $\theta$; it does not touch $\theta$ at all.
- **Adjacent** ($a$) — the side that touches $\theta$ but is *not* the hypotenuse.

Which side counts as "opposite" or "adjacent" depends entirely on *which angle* you are


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measuring from — if you moved to the other non-right angle, the two labels would swap.

Because this is a right triangle, the three sides always satisfy the **Pythagorean Theorem**:

$$a^2 + b^2 = h^2$$

## Trigonometric Identities

Before we can prove the Sine, Cosine, and Projection formulas, we need three families of identities that let us rewrite sin, cos, and tan in different, equally valid forms. These are not new facts to memorise separately — they all fall directly out of the right-triangle definitions above.

### Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

WHY THIS WORKS

csc, sec, and cot are simply the *reciprocals* ("flip the fraction") of sin, cos, and tan. If $\sin\theta = \frac{b}{h}$, then by definition $\csc\theta = \frac{h}{b}$ — which is exactly $\frac{1}{\sin\theta}$. The same logic gives $\sec\theta$ from $\cos\theta$, and $\cot\theta$ from $\tan\theta$.

### Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sec^2\theta = 1 + \tan^2\theta \qquad \csc^2\theta = 1 + \cot^2\theta$$

WHY THIS WORKS

These three identities are all restatements of $a^2 + b^2 = h^2$ from the previous page. Dividing every term of $a^2 + b^2 = h^2$ by $h^2$ gives $\sin^2\theta + \cos^2\theta = 1$; dividing instead by $a^2$ or by $b^2$ gives the other two. They are extremely useful for simplifying expressions and for switching between sin, cos, and tan inside a proof — you will see $\sin^2\theta + \cos^2\theta = 1$ used directly in the Cosine Rule proof later in this unit.

### Half-Angle Formulas

$$\sin\left(\tfrac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} \qquad \cos\left(\tfrac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}} \qquad \tan\left(\tfrac{\theta}{2}\right) = \frac{1-\cos\theta}{\sin\theta}$$


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NOTE

These let you find sin, cos, or tan of *half* an angle when you already know cos (and sin, for the tan case) of the full angle. The $\pm$ sign on the first two is there because the half-angle could land in a quadrant where that ratio is negative — you choose + or − based on which quadrant $\frac{\theta}{2}$ actually falls in. The $\tan\left(\frac{\theta}{2}\right)$ form needs no $\pm$, since dividing $1-\cos\theta$ by $\sin\theta$ already carries the correct sign automatically.

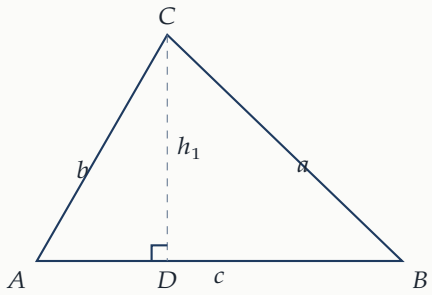
## 1. Law of Sine

### Statement

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

IN PLAIN WORDS

In *any* triangle — not just a right-angled one — if you divide each side by the sine of the angle directly opposite it, you always get the same number, no matter which of the three sides you pick. This is what makes the Sine Rule so useful: it connects every side to every angle in one single equation.




*Triangle ABC with altitude $CD = h_1$ dropped from vertex C onto side AB, splitting it into two right triangles, ACD and BCD. This altitude is the key construction the whole proof depends on.*

PROOF

Recall that $\sin A = \dfrac{\text{opposite}}{\text{hypotenuse}}$.

Let $CD = h_1$. Then in right triangle $ACD$:

$$\sin A = \frac{h_1}{b} \implies b\sin A = h_1$$

And in right triangle $BCD$:

$$\sin B = \frac{h_1}{a} \implies a\sin B = h_1$$

We know that $h_1 = h_1$, so:

$$b\sin A = a\sin B$$

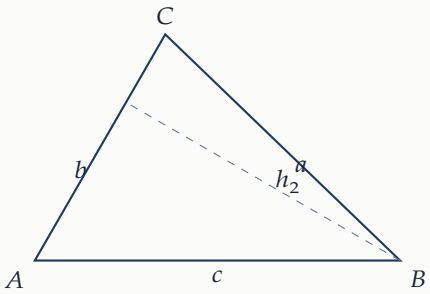

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$$\frac{b}{\sin B} = \frac{a}{\sin A} \qquad \text{—(1)}$$

WHAT JUST HAPPENED?

The altitude $CD$ was measured two different ways — once from angle $A$'s triangle, once from angle $B$'s triangle — and both had to give the same length, $h_1$. Setting those two expressions equal to each other is the whole trick behind this proof: it is what links side $a$ and side $b$ to their opposite angles in the same equation.

Now the same trick is repeated with a *second* altitude, $h_2$, dropped from vertex $B$ onto line $AC$, to bring angle $C$ into the picture too:




*The second altitude, $h_2$, dropped from vertex B perpendicular onto line AC.*

PROOF (CONTINUED)

In right triangle $BEC$ (where $E$ is the foot of $h_2$ on $AC$):

$$\sin C = \frac{h_2}{a} \implies a \sin C = h_2$$

And in right triangle $BEA$:

$$\sin A = \frac{h_2}{c} \implies c \sin A = h_2$$

We know that $h_2 = h_2$, so:

$$a \sin C = c \sin A$$

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \text{—(2)}$$

Using (1) and (2) together:

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

Hence the proof. ■

PUTTING IT TOGETHER

Step (1) linked sides $a$ and $b$ to angles $A$ and $B$ using the first altitude. Step (2) linked sides $a$ and $c$ to angles $A$ and $C$ using the second altitude. Because both steps share the ratio $\frac{a}{\sin A}$, chaining them together gives all three ratios equal at once — which is exactly the Sine Rule.


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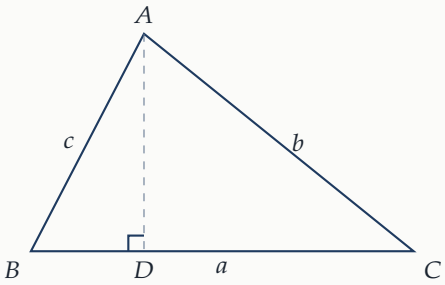
## 2. Law of Cosine

### Statement

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

IN PLAIN WORDS

The Cosine Rule connects all three sides of a triangle to just *one* of its angles. It is what you reach for whenever the Sine Rule cannot be used directly — for example, when you know all three sides but no angles at all, or when you know two sides and the angle trapped between them. Notice the pattern: each formula puts the squared side opposite the angle you are solving for by itself in the numerator, subtracted from the sum of the other two squared sides.




*Triangle ABC with altitude AD dropped from vertex A onto side BC = a. This is the same style of construction used to prove the Sine Rule — an altitude that creates two right triangles to work with.*

PROOF (derivation of $\cos C$)

In right triangle $ABD$, by Pythagoras:

$$AB^2 = BD^2 + AD^2$$

In right triangle $ADC$:

$$\sin C = \frac{AD}{AC} \implies AC \sin C = AD \qquad —(1)$$

$$b \sin C = AD$$

Also in right triangle $ADC$:

$$\cos C = \frac{DC}{AC} \implies AC \cos C = DC$$

Since $BD = a - DC$:

$$BD = a - AC \cos C = a - b \cos C$$


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WHY TWO RIGHT TRIANGLES?

The altitude $AD$ splits triangle $ABC$ into two right triangles, $ABD$ and $ADC$, exactly like the small SOH-CAH-TOA sketches at the top of this page show. Triangle $ADC$ (which contains the known angle $C$) is used first to express both $AD$ and $DC$ in terms of side $b = AC$ and angle $C$. Those expressions will be substituted into the Pythagoras equation for the *other* right triangle, $ABD$, on the next page — that substitution is what produces the Cosine Rule.

PROOF (CONTINUED)

Now substitute $BD = a - b\cos C$ and $AD = b\sin C$ into the Pythagoras equation for right triangle $ABD$:

$$AB^2 = BD^2 + AD^2$$

$$c^2 = (a - b\cos C)^2 + (b\sin C)^2$$

Expanding the brackets:

$$c^2 = a^2 - 2ab\cos C + b^2\cos^2 C + b^2\sin^2 C$$

$$c^2 = a^2 + b^2(\cos^2 C + \sin^2 C) - 2ab\cos C$$

Using the Pythagorean identity $\cos^2 C + \sin^2 C = 1$ from earlier:

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Rearranging to make $\cos C$ the subject:

$$2ab\cos C = a^2 + b^2 - c^2$$

$$\boxed{\cos C = \frac{a^2 + b^2 - c^2}{2ab}}$$

Hence the proof. ■

*The relations for* $\cos A$ *and* $\cos B$ *follow identically, by dropping the altitude from B or C instead.*

WHY THIS WORKS

The expansion of $(a - b\cos C)^2$ uses the same algebraic pattern as $(x - y)^2 = x^2 - 2xy + y^2$. The key move is grouping $b^2\cos^2 C + b^2\sin^2 C$ together as $b^2(\cos^2 C + \sin^2 C)$, which collapses to just $b^2$ using the Pythagorean identity — this is exactly why that identity was introduced at the start of the unit: it is what turns a messy expansion into the clean Cosine Rule.


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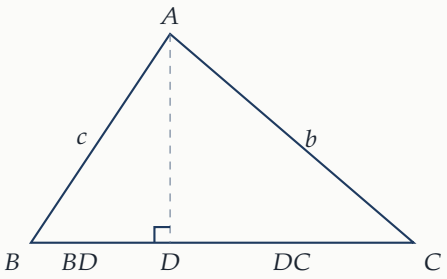
## 3. Projection Formula

### Statement

$$a = c\cos B + b\cos C \qquad b = a\cos C + c\cos A \qquad c = a\cos B + b\cos A$$

IN PLAIN WORDS

Each side of a triangle can be rebuilt out of the other two sides' *projections* onto it — literally, the "shadow" each side would cast onto it at a right angle. This is why it's called the Projection Formula: it expresses a side as the sum of two shorter projected lengths rather than relating it to areas or ratios like the Sine and Cosine Rules do.




*Side $a = BC$ is made up of the two projections $BD$ and $DC$, created by dropping the altitude $AD$ from $A$.*

PROOF (derivation of $a = c\cos B + b\cos C$)

Since $BC = a = BD + DC$:

$$a = BD + DC$$

Multiply and divide each term by the hypotenuse of its own right triangle — *exchange* $AD$ for $AB$ in the first term, and keep $AC$ in the second:

$$a = \frac{BD}{AB}\cdot AB + \frac{DC}{AC}\cdot AC$$

WHY MULTIPLY AND DIVIDE?

Multiplying $BD$ by $\frac{AB}{AB}$ (which is just 1) doesn't change its value — but it rewrites it as $\frac{BD}{AB}\cdot AB$, and $\frac{BD}{AB}$ is recognisable as $\cos B$ from right triangle $ABD$. This "multiply by a clever form of 1" trick is what lets the proof swap from raw lengths into cosines in the next step.

PROOF (CONTINUED)

In right triangle $ABD$: $\cos B = \frac{BD}{AB} \implies BD = AB\cos B = c\cos B$

In right triangle $ADC$: $\cos C = \frac{DC}{AC} \implies DC = AC\cos C = b\cos C$


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Substituting back into $a = \frac{BD}{AB} \cdot AB + \frac{DC}{AC} \cdot AC$:

$$\boxed{a = c\cos B + b\cos C}$$

Hence the proof. ■
*Dropping the altitude from the other two vertices gives $b = a\cos C + c\cos A$ and $c = a\cos B + b\cos A$ by the same method.*

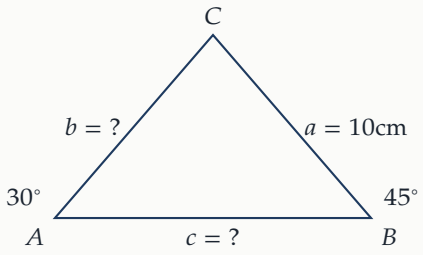
This completes all three formulas from this unit's learning objectives, each with a full proof. The rest of this unit applies them to worked examples.

## Worked Examples — Sine Rule

**Example 1**

In a triangle, $A = 30°$, $B = 45°$, and side $a = 10$ cm. Find sides $b$ and $c$.
*Solution.* By using the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$




$$\frac{10}{\sin 30°} = \frac{b}{\sin 45°}$$
$$\frac{10}{0.5} = \frac{b}{0.707}$$
$$b = \frac{10 \times 0.707}{0.5}$$
$$\boxed{b = 14.14 \text{ cm}}$$

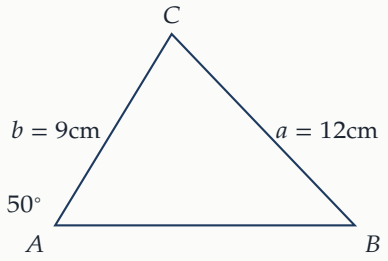
Then, to find side $c$: since the three angles of a triangle sum to 180°,

$$C = 180° - 30° - 45° = 105°$$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \implies \frac{10}{\sin 30°} = \frac{c}{\sin 105°}$$
$$c = \frac{10 \times \sin 105°}{\sin 30°} = \frac{10 \times 0.966}{0.5}$$
$$\boxed{c = 19.32 \text{ cm}}$$


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**Example 2**

In a triangle, side $a = 12$ cm, $b = 9$ cm, and angle $A = 50°$. Find angle $B$.




*Solution.* From the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{12\text{ cm}}{\sin 50°} = \frac{9\text{ cm}}{\sin B}$$

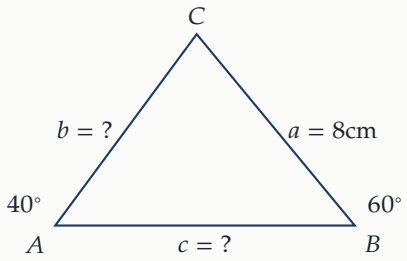
$$\sin B = \frac{9\text{ cm} \times \sin 50°}{12\text{ cm}} = \frac{9 \times 0.766}{12} = \frac{6.894}{12}$$

$$B = \sin^{-1}(0.574)$$

$$\boxed{B \approx 35° \text{ (approximation)}}$$

**Example 3**

In triangle $ABC$, $A = 40°$, $B = 60°$, $a = 8$ cm. Find $b$.




*Solution.* First find angle $C$: since the angles of a triangle sum to 180°,

$$A + B + C = 180°$$

$$40° + 60° + C = 180°$$

$$C = 180° - 100° = 80°$$

Then, from the sine rule, find side $b$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \implies \frac{8\text{ cm}}{\sin 40°} = \frac{b}{\sin 60°}$$

$$b = \frac{8\text{ cm} \times \sin 60°}{\sin 40°} = \frac{8 \times 0.866}{0.643}$$


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$$\boxed{b = 10.78 \text{ cm}}$$

Finally, find side $c$ the same way:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \implies \frac{8 \text{ cm}}{\sin 40^\circ} = \frac{c}{\sin 80^\circ}$$

$$c = \frac{8 \text{ cm} \times \sin 80^\circ}{\sin 40^\circ} = \frac{8 \times 0.985}{0.643}$$

$$\boxed{c = 12.26 \text{ cm}}$$

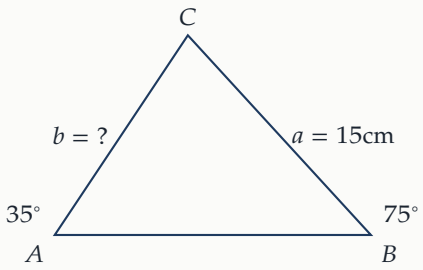
## Home Work

TRY THESE YOURSELF

1. In a triangle, $a = 15$ cm, $A = 35^\circ$, $B = 75^\circ$. Find side $b$.
2. In a triangle, $b = 10$ cm, $c = 14$ cm, and angle $B = 55^\circ$. Find angle $C$.

**Home Work 1**

*Solution.*




From the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{15 \text{ cm}}{\sin 35^\circ} = \frac{b}{\sin 75^\circ}$$

$$b = \frac{15 \text{ cm} \times \sin 75^\circ}{\sin 35^\circ} = \frac{15 \text{ cm} \times 0.966}{0.574}$$

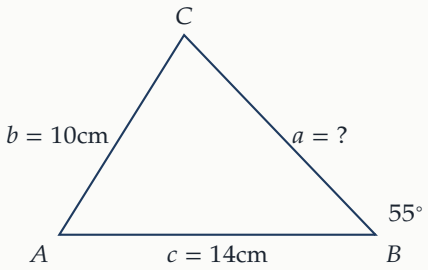
$$\boxed{b = 25.24 \text{ cm}}$$

The length of $b = 25.24$ cm.

**Home Work 2**

*Solution.*


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From the sine rule:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{10\text{ cm}}{\sin 55^\circ} = \frac{14\text{ cm}}{\sin C} \implies \frac{10\text{ cm}}{0.819} = \frac{14\text{ cm}}{\sin C}$$

$$\sin C = \frac{0.819 \times 14\text{ cm}}{10\text{ cm}} = \frac{11.466\text{ cm}}{10\text{ cm}}$$

$$\sin C = 1.147$$

$$C = \sin^{-1}(1.147)$$

$$C = 30.67^\circ$$

Angle $C = 30.67^\circ$.

IMPORTANT: ALWAYS CHECK YOUR ANSWER!

Notice that $\sin C = 1.147$ here — but sin of any real angle can never be larger than 1. This is a warning sign, not just a rounding slip: it means the three numbers given in this question ($b = 10$ cm, $c = 14$ cm, $B = 55^\circ$) cannot actually form a real triangle together. Whenever a calculation gives you $\sin\theta > 1$ or $\sin\theta < -1$, stop and re-check the numbers you were given, rather than trusting the calculator's $\sin^{-1}$ output — most calculators will still return a value (often using complex numbers behind the scenes) even when the answer has no real-world meaning.

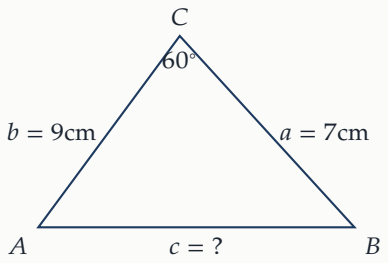

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## Worked Examples — Cosine Rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

**Example 1**

In triangle $ABC$, $a = 7$ cm, $b = 9$ cm, and included angle $C = 60°$. Find side $c$.




*Solution.* From the cosine rule:

$$\cos 60° = \frac{(9)^2 + (7)^2 - c^2}{2 \times 9 \times 7}$$

$$0.5 = \frac{81 + 49 - c^2}{126}$$

$$0.5 \times 126 = 81 + 49 - c^2$$

$$63 = 130 - c^2$$

$$c^2 = 130 - 63 = 67$$

$$\sqrt{c^2} = \sqrt{67}$$

$$\boxed{c = 8.19 \text{ cm}}$$

**Example 2**

In triangle $ABC$, $a = 12$ cm, $b = 16$ cm, $c = 20$ cm. Find angle $C$.

*Solution.* From the formula:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{(12\text{cm})^2 + (16\text{cm})^2 - (20\text{cm})^2}{2 \times 12\text{cm} \times 16\text{cm}}$$

$$\cos C = \frac{144\text{cm} + 256\text{cm} - 400\text{cm}}{384\text{cm}}$$

$$\cos C = \frac{0}{384\text{cm}}$$

$$C = \cos^{-1}(0)$$

$$\boxed{C = 90°}$$


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A USEFUL SHORTCUT

Whenever $\cos C = 0$ comes out of the Cosine Rule, the angle is automatically $90°$ — which is really just the Cosine Rule confirming the Pythagorean Theorem: when $a^2 + b^2 = c^2$ (as it does here: $144 + 256 = 400$), the triangle is right-angled at $C$. This is a good way to double-check whether a triangle is right-angled once you know all three sides.

END OF UNIT I — PROPERTIES OF TRIANGLE

Unit II — Heron's Formula for the Area of a Triangle — follows next.


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