UNIT 1

MATRICES AND DETERMINANTS

How a square matrix reduces to a single number, and how that number is read.

1.1 Definition — the Determinant

DETERMINANT

a. Is a **scalar-valued function** of the entries of a square matrix; or

b. Is a **single numeric value** that represents a matrix.

Denoted as **|A|** or **det(A)** — the two notations are interchangeable.

1.2 Matrix versus Determinant

THREE POINTS OF DIFFERENCE

Determinant	**Matrix**
i. Expresses a **numerical value**.	i. Does not give a numerical value — it is an array of entries.
ii. Always belongs to a **square matrix**.	ii. Is not restricted to square form.
iii. Represented as **|A|**.	iii. Represented as [A].

1.3 Properties of the Determinant

Property I — Interchanging the rows and columns of a matrix (its **transpose**) does not change the arrangement's determinant value.

$$\boldsymbol{B} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{ROWS} \leftrightarrow \text{COLUMNS}$$

Written as determinants, the two sides carry the **same value**:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property II — Interchanging **any two** rows or columns reverses the sign of the determinant.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{rows 1 \& 2 swapped}$$

Property III — If two rows or two columns are equal, **or** any single row or column is entirely zero, the determinant equals zero.

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{vmatrix} = \boldsymbol{0}$$

ASSIGNMENT

1. Find the determinant of each matrix below.

$$A = \begin{bmatrix} 4 & 6 \\ 1 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 1 & 3 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 4 \end{bmatrix}$$

2. Construct a square matrix of your own choice. Using that matrix, show that it satisfies **Property I** (transpose), **Property II** (row/column swap) and **Property III** (equal or zero rows) from Section 1.3.

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1.4 What's Ahead

UNIT 1 · CONTINUED

→ **Inverse of a matrix**

* Transpose
* Cramer's rule

UNIT 2 · PREVIEW

* Vector — definition and explanation
* Vector operations and properties
* Vector equations in a given coordinate system

1.5 Inverse of a Matrix

Worked example — suppose:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ find } \boldsymbol{A^{-1}}$$

STEP 1 — GENERAL FORMULA

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

STEP 2 — EVALUATE THE DETERMINANT

$$\det(A) = |A| = (1 \times 4) - (-3 \times -2) = 4 - 6 = \mathbf{-2}$$

STEP 3 — SUBSTITUTE BACK

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} \frac{-4}{2} & \frac{-2}{-2} \\ \frac{-3}{2} & \frac{-1}{2} \end{bmatrix}$$

STEP 4 — SIMPLIFY EACH ENTRY

$$\begin{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

RESULT **A^{-1} exists, since det(A) = −2 ≠ 0**

1.6 Transpose of a Matrix

The transpose of a matrix is formed by turning its **rows into columns** (and columns into rows), keeping every entry in place along the way.

Suppose:

$$\boldsymbol{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$$

then its transpose, written $\mathbf{B^T}$, is:

$$\boldsymbol{B^T} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}$$ COLUMN 1 OF B → ROW 1 OF B^T

1.7 Cramer's Rule

Consider a system of linear equations, represented in the form:

$$\boldsymbol{A}\ \mathbf{x} = \boldsymbol{B}$$

WHERE

A is the **coefficient matrix**, A = { a_1, a_2, … }

x is the **variable vector**, x = { x_1, x_2, x_3, … }

B is the vector of **constant terms**, B = { b_1, b_2, … }

Written out in full, a system of **n** equations takes the form:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 = b_n$$

Cramer's rule finds each variable as the ratio of two determinants — the coefficient determinant, and a version of it with one column replaced by the constants column:

$$\boldsymbol{x_1} = \frac{|A_{x1}|}{|A|} \quad , \quad \boldsymbol{x_2} = \frac{|A_{x2}|}{|A|}$$

$$\boldsymbol{x_3} = \frac{|A_{x3}|}{|A|} \quad , \quad \boldsymbol{x_n} = \frac{|A_{xn}|}{|A|}$$

Example — P is a linear system of order 3. Find the values of y_1, y_2 and y_3 in the system below:

$$P \begin{cases} y_1 + 2y_2 + 4y_3 = 6 \\ 6y_1 + y_2 \quad\quad = 1 \\ 2y_1 + 5y_2 + 2y_3 = 5 \end{cases}$$

SOLUTION

STEP 1 – REPRESENT THE LINEAR SYSTEM AS A MATRIX

$$\boldsymbol{P} = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 1 & 0 \\ 2 & 5 & 2 \end{bmatrix}$$

STEP 2 – WRITE IT AS AX = B

$$\begin{bmatrix} 1 & 2 & 4 \\ 6 & 1 & 0 \\ 2 & 5 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

STEP 3 — FIND THE DETERMINANT OF SYSTEM P

$$\boldsymbol{P} = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 1 & 0 \\ 2 & 5 & 2 \end{bmatrix}$$

$$= 1\times \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} - 2\times \begin{vmatrix} 6 & 0 \\ 2 & 2 \end{vmatrix} + 4\times \begin{vmatrix} 6 & 1 \\ 2 & 5 \end{vmatrix}$$

$$= 1(2-0) - 2(12-0) + 4(30-2)$$

$$|P| \quad \mathbf{90}$$

STEP 4 — REPLACE THE Y_1 COLUMN WITH B, THEN FIND $|P_{Y1}|$

$$|\boldsymbol{P_{y1}}| = \begin{vmatrix} 6 & 2 & 4 \\ 1 & 1 & 0 \\ 5 & 5 & 2 \end{vmatrix}$$

$$= 6\times \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} - 2\times \begin{vmatrix} 1 & 0 \\ 5 & 2 \end{vmatrix} + 4\times \begin{vmatrix} 1 & 1 \\ 5 & 5 \end{vmatrix}$$

$$= 6(2-0) - 2(2-0) + 4(5-5) = 12 - 4$$

$$|P_{Y1}| \quad \mathbf{8}$$

STEP 5 — APPLY CRAMER'S RULE

$$\boldsymbol{y_1} = \frac{|P_{y1}|}{|P|} = \frac{8}{90}$$

STEP 6 — REPLACE THE Y_2 COLUMN WITH B, THEN FIND $|P_{Y2}|$

$$\begin{vmatrix} & \\ & \end{vmatrix}$$

$$|P_{y2}| = \begin{vmatrix} 1 & 6 & 4 \\ 6 & 1 & 0 \\ 2 & 5 & 2 \end{vmatrix}$$

$$= 1(2-0) - 6(12-0) + 4(30-2) = 2 - 72 + 112$$

$|P_{Y2}|$ **42**

$$y_2 = \frac{|P_{y2}|}{|P|} = \frac{42}{90}$$

STEP 7 — REPLACE THE Y_3 COLUMN WITH B, THEN FIND $|P_{Y3}|$

$$|P_{y3}| = \begin{vmatrix} 1 & 2 & 6 \\ 6 & 1 & 1 \\ 2 & 5 & 5 \end{vmatrix}$$

$$= 1(5-5) - 2(30-2) + 6(30-2)$$

$$= -56 + 168$$

$|P_{Y3}|$ **112**

$$y_3 = \frac{|P_{y3}|}{|P|} = \frac{112}{90}$$

ANSWER $\mathbf{y_1 = 8/90 \cdot y_2 = 42/90 \cdot y_3 = 112/90}$